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Iterative solution of a coupled mixed and standard Galerkin discretization method for elliptic problems

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SUMMARY

In this paper, we consider approximation of a second-order elliptic problem defined on a domain in two-dimensional Euclidean space. Partitioning the domain into two subdomains, we consider a technique proposed by Wieners and Wohlmuth [9] for coupling mixed finite element approximation on one subdomain with a standard finite element approximation on the other. In this paper, we study the iterative solution of the resulting linear system of equations. This system is symmetric and indefinite (of saddle-point type). The stability estimates for the discretization imply that the algebraic system can be preconditioned by a block diagonal operator involving a preconditioner for $\mathbf{H}(\mathrm{div})$ (on the mixed side) and one for the discrete Laplacian (on the finite element side). Alternatively, we provide iterative techniques based on domain decomposition. Utilizing subdomain solvers, the composite problem is reduced to a problem defined only on the interface between the two subdomains. We prove that the interface problem is symmetric, positive definite and well conditioned and hence can be effectively solved by a conjugate gradient iteration. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: combined mixed and standard Galerkin discretization methods; non-matching grids; preconditioning

1. INTRODUCTION

One of the main problems in large-scale scientific computation is the time required to set up a problem. In applications which involve partial differential equations on complicated domains, a great deal of effort is required to construct the mesh. Often, complex domains are built up

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from simpler ones. The mesh construction problem is greatly simplified if the simpler domains (i.e., the subdomains) can be meshed independently. This, however, results in meshes which do not align on the internal interfaces between subdomains. To get accurate approximation with such meshes, various techniques have been developed.

Since meshes do not align, the resulting spaces are necessarily non-conforming. Approximate continuity conditions are imposed by the use of a Lagrange multiplier [1–7]. There are two approaches for the analysis of the composite problem. The first treats the method as a non-conforming finite element approximation where the Lagrange multiplier constraints serve to define the non-conforming approximation subspace. The second approach is based on an appropriate Ladyzhenskaya–Babuška–Brezzi (LBB) condition. With the second approach, the discrete Lagrange multiplier is shown to approximate the continuous Lagrange multiplier, often a quantity of physical interest. In both cases, the Lagrange multiplier space needs to be strongly connected to the approximation in the subdomains. For the mortar finite element approximation, this connection comes from defining the Lagrange multiplier space using the mesh on one of the subdomains [3]. For the LBB condition, one is often required to use a multiplier space on a mesh which is somewhat coarser than the mesh on the subdomains [1, 5, 6, 8].

We consider an approximation technique proposed in Reference [9] which utilizes a finite element discretization on one subdomain and a mixed finite element discretization on the other. Such a situation may arise when one wants to couple two existing implementations for numerical simulation of different processes (say diffusion in one domain and convection–reaction–diffusion in the other) which interact through a common boundary Γ. This approximation gives rise to a natural variational reformulation of the original problem into a saddle point problem involving the two unknown functions (velocity/pressure) on the mixed side and the original unknown function (pressure) on the conforming finite element side. No additional multipliers need to be introduced.

We focus, in the present paper, on the iterative solution of the coupled problem. Methods for solving linear systems resulting from finite element approximations of elliptic problems on non-matching grids have been subjected to numerous studies (see, e.g., References [10–16]). The main goal of these works is to find a preconditioner for the system which is spectrally equivalent to the matrix of the problem, uniformly with respect to the number of subdomains. For more references concerning domain decomposition methods, including discretization methods on non-matching grids we refer to the Proceedings of the International Symposia on Domain Decomposition Methods [17–20].

The goal of this paper is slightly different from the goals of the works mentioned above. Namely, we want to develop iterative methods for the solution of the coupled Galerkin/mixed finite element system of algebraic equations in the case where there are just two subdomains and in one of the subdomains the Galerkin method has been used while in the other subdomain a mixed finite element approximation is employed. Our analysis does not exclude the case when in the subdomain of mixed or/and Galerkin approximations one uses non-matching grids and mortar approximations. However, we do not pursue this direction in the present work.

The analysis of the coupled approximation is based on a stability estimate for the coupled mixed system. It follows from this estimate that it is possible to precondition the full system if preconditioners for $\mathbf{H}(\mathrm{div})$ (on the mixed finite element subdomain) and H^1 (on the Galerkin finite element subdomain) are available. Alternatively, the domain decomposition algorithms which we study require the solution of mixed and Galerkin finite element subproblems on each subdomain and reduce the problem to one on the interface between the two subdomains. We consider two algorithms of this type. The first iterates for the trace of the discrete solution on the interface while the second iterates for the trace of the discrete normal derivative on the interface. Both algorithms can be thought of as Neumann–Dirichlet maps in that the discrete subproblems correspond to problems with Neumann and Dirichlet boundary conditions on their respective subdomains.

The outline of the remainder of the paper is as follows. In Section 2 we introduce the coupled mixed and conforming variational formulation of the original problem. In Section 3 we propose and study the stability and derive an error estimate for the corresponding finite element discretization. Several iterative solution methods for the resulting system of algebraic equations are considered in Section 4. Finally, Section 5 gives numerical results which illustrate the theory given in the earlier sections.

2. VARIATIONAL FORMULATION

Consider the model second-order elliptic problem on a domain Ω contained in $\mathbf{R}^2$

$$\mathscr{L}p \equiv -\nabla \cdot a\nabla p = f(x), \quad x \in \Omega \tag{1}$$

with, for example, homogeneous Dirichlet boundary conditions $p=0$ on $\partial\Omega$. Here $a(x)$ is symmetric and uniform in Ω positive-definite 2×2 matrix with piece-wise smooth elements, i.e. there is a constant $a_0>0$ such that the following inequality is valid uniformly in Ω:

$$a_0|\underline{\xi}|^2 \leqslant \underline{\xi}^{\mathrm{T}} a \underline{\xi}, \quad \forall \underline{\xi} \in \mathbf{R}^2$$

With some abuse of the terminology we shall call the solutions of the equation $\mathscr{L}q=0$ harmonic functions. Of course, if a is the identity matrix, then q is harmonic in Ω.

We partition Ω into two subdomains by an interface boundary Γ, i.e. let $\Omega=\Omega_1\cup\Gamma\cup\Omega_2$ (see Figure 1). In Ω_1 we will use a mixed setting of problem (1). That is, we introduce the new (vector) variable $\mathbf{u}=-a\nabla p$ so that Equation (1) reduces to $\nabla\cdot\mathbf{u}=f$. To distinguish

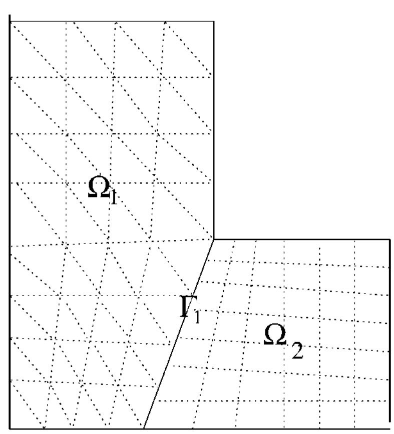

Figure 1. Domain partitioning of $\Omega=\Omega_1\cup\Gamma\cup\Omega_2$.

between the problem settings we will write $p_1 = p|_{\Omega_1}$ and $p_2 = p|_{\Omega_2}$. Thus, Equation (1) decomposes into two different problems in Ω_1 and Ω_2, correspondingly

$$a^{-1}\mathbf{u} = -\nabla p,\ \ \nabla \cdot \mathbf{u} = f(x),\ \ x \in \Omega_1,\ \ -\nabla \cdot a\nabla p = f(x),\ \ x \in \Omega_2$$

coupled through the interface conditions on Γ: $p_1(x) = p_2(x), -a\nabla p_2 \cdot \mathbf{n}_1 = \mathbf{u} \cdot \mathbf{n}_1$. Here $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit normal vectors to Γ pointing outward to Ω_1 and Ω_2, respectively.

Next, we consider the spaces $\mathbf{H}(\mathrm{div})$, L^2 and H^1 over the domains Ω_1 and Ω_2 and introduce the notations:

$$\mathbf{H}(\mathrm{div}, \Omega_1) \equiv \mathbf{V}, \quad L^2(\Omega_1) \equiv Q_1$$

and

$$H_0^1(\Omega_2, \partial\Omega_2 \backslash \Gamma) = \{\phi \in H^1(\Omega_2);\ \ \phi = 0 \ \text{ on } \partial\Omega_2 \backslash \Gamma\} \equiv Q_2$$

We will denote $\|\cdot\|_{\mathbf{V}}$ to be the $\mathbf{H}(\mathrm{div})$ norm on $\mathbf{V}$. Further, we will use the following additional notation:

$$\langle p, q\rangle_\Gamma = \int_\Gamma pq \, \mathrm{d}s \quad \text{and} \quad a(p_2, q_2) = \int_{\Omega_2} a\nabla p_2 \cdot \nabla q_2 \, \mathrm{d}x \tag{2}$$

Whenever there is no ambiguity we will use $(\cdot,\cdot)$ to denote the L^2-inner product with respect to a domain (mostly Ω_1 or Ω_2). We will also use $H^s(\Omega)$ to denote the Sobolev space on Ω of order s (see, for example, References [21, 22]). We set $\mathbf{H}^s(\Omega_1) = H^s(\Omega_1)^2$. The corresponding norm will be denoted by $\|\cdot\|_{s,\Omega}$. When s is not integer these spaces are generated by the real interpolation method (see, e.g., References [21–23]).

Testing the equation $a^{-1}\mathbf{u} + \nabla p_1 = 0$ by a function $\underline{\chi} \in \mathbf{V}$, integrating by parts, using the zero boundary conditions for p_1 on $\partial\Omega_1 \backslash \Gamma$ and the fact that p_1 is equal to p_2 on Γ, we get

$$(a^{-1}\mathbf{u},\ \underline{\chi}) - (p_1, \nabla \cdot \underline{\chi}) + \langle p_2, \underline{\chi} \cdot \mathbf{n}_1\rangle_\Gamma = 0 \quad \text{for all } \underline{\chi} \in \mathbf{V} \tag{3}$$

The second equation is obtained by testing $\nabla \cdot \mathbf{u} = f$ on Ω_1 by a function $q_1 \in Q_1$:

$$-(\nabla \cdot \mathbf{u}, q_1) = -(f, q_1) \quad \text{for all } q_1 \in Q_1 \tag{4}$$

Finally, testing the original Equation (1) by a function $q_2 \in Q_2$, integrating by parts, using the zero boundary condition for q_2 on $\partial\Omega_2 \backslash \Gamma$ and the fact that $\mathbf{u} \cdot \mathbf{n}_1 = -a\nabla p_1 \cdot \mathbf{n}_1 = a\nabla p_2 \cdot \mathbf{n}_2$ on Γ gives

$$\langle \mathbf{u} \cdot \mathbf{n}_1, q_2\rangle_\Gamma - a(p_2, q_2) = -(f, q_2), \quad \text{for all } q_2 \in Q_2 \tag{5}$$

Thus, the three unknowns $(\mathbf{u}, p_1, p_2) \in \mathbf{V} \times Q_1 \times Q_2$ satisfy the coupled system

$$\begin{aligned} (a^{-1}\mathbf{u},\ \underline{\chi}) - (p_1, \nabla \cdot \underline{\chi}) + \langle p_2, \underline{\chi} \cdot \mathbf{n}_1,\rangle_\Gamma &= 0 \quad \text{for } \underline{\chi} \in \mathbf{V} \\ -(\nabla \cdot \mathbf{u}, q_1) &= -(f, q_1) \quad \text{for } q_1 \in Q_1, \\ \langle \mathbf{u} \cdot \mathbf{n}_1, q_2\rangle_\Gamma - a(p_2, q_2) &= -(f, q_2) \quad \text{for } q_2 \in Q_2 \end{aligned} \tag{6}$$

2.1. Well-posedness of the composite problem

Following [9], we reorder the unknowns and consider the generalized system

$$\begin{aligned} (a^{-1}\mathbf{u},\ \underline{\chi}) + \langle p_2, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma - (p_1, \nabla\cdot\underline{\chi}) &= \langle \mathbf{F}_0, \underline{\chi}\rangle \quad \text{for } \underline{\chi}\in\mathbf{V} \\ -\langle \mathbf{u}\cdot\mathbf{n}_1, q_2\rangle_\Gamma + a(p_2, q_2) &= \langle F_2, q_2\rangle \quad \text{for } q_2\in Q_2 \\ -(\nabla\cdot\mathbf{u}, q_1) &= \langle F_1, q_1\rangle \quad \text{for } q_1\in Q_1 \end{aligned} \tag{7}$$

Here $\mathbf{F}_0, F_1$, and F_2 are elements of the spaces $\mathbf{V}'$, Q_1', and Q_2' of bounded linear functionals in $\mathbf{V}$, Q_1, and Q_2, respectively. Finally, $\langle\cdot,\cdot\rangle$ denotes the pairing between a space and its dual.

The analysis of the above problem (done essentially in Reference [9]) is based on considering it as a block saddle-point problem of the form

$$\begin{aligned} \mathbf{A}(\mathbf{u}, p_2) + \mathbf{B}^{\mathrm{T}} p_1 &= \tilde{F}_1 \\ \mathbf{B}(\mathbf{u}, p_2) &= \tilde{F}_2 \end{aligned} \tag{8}$$

Here

$$\mathbf{A}: \mathbf{V}\times Q_2 \to (\mathbf{V}\times Q_2)', \quad \mathbf{B}: \mathbf{V}\times Q_2 \to Q_1', \quad \mathbf{B}^{\mathrm{T}}: Q_1 \to (\mathbf{V}\times Q_2)'$$

are defined for $\mathbf{u},\mathbf{v}\in\mathbf{V}$, $p_2, r_2\in Q_2$ and $p_1, r_1\in Q_1$ by

$$\begin{aligned} \langle \mathbf{A}(\mathbf{u}, p_2), (\mathbf{v}, r_2)\rangle &= (a^{-1}\mathbf{u}, \mathbf{v}) + \langle \mathbf{v}\cdot\mathbf{n}_1, p_2\rangle_\Gamma - \langle \mathbf{u}\cdot\mathbf{n}_1, r_2\rangle_\Gamma + a(p_2, r_2) \\ \langle \mathbf{B}(\mathbf{u}, p_2), r_1\rangle &= \langle \mathbf{B}^T r_1, (\mathbf{u}, p_2)\rangle = -(r_1, \nabla\cdot\mathbf{u}) \end{aligned} \tag{9}$$

Clearly, $\operatorname{Ker}\mathbf{B} = \{(\mathbf{u}, p_2): \nabla\cdot\mathbf{u} = 0\}$. It immediately follows that $\mathbf{A}$ is coercive on $\operatorname{Ker}\mathbf{B}$. Moreover, the 'inf-sup' condition corresponding to (9) is the standard condition for the mixed method on Ω_1 alone, namely, there is a constant $C>0$, such that

$$C\|q_1\|_{0,\Omega_1} \leqslant \sup_{\underline{\chi}\in\mathbf{V}} \frac{(q_1, \nabla\cdot\underline{\chi})}{\|\underline{\chi}\|_{\mathbf{V}}} \quad \text{for all } q_1\in Q_1$$

Then the following theorem is an immediate consequence (see Theorem 1.1 in Reference [24, p. 42]).

Theorem 2.1. *There exists exactly one solution* $(\mathbf{u}, p_1, p_2)$ *of* (7) *in* $\mathbf{V}\times Q_1\times Q_2$. *Moreover, there is a constant* C *not depending on* $F_0\in\mathbf{V}'$, $F_1\in Q_1'$ *and* $F_2\in Q_2'$ *such that*

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p_1\|_{0,\Omega_1} + \|p_2\|_{1,\Omega_2} \leqslant C(\|\mathbf{F}_0\|_{\mathbf{V}'} + \|F_1\|_{Q_1'} + \|F_2\|_{Q_2'})$$

3. FINITE ELEMENT DISCRETIZATION

In this section, we present the finite element discretization of problem (7). Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be triangulations of Ω_1 and Ω_2. We assume that both triangulations $\mathscr{T}_1$ and $\mathscr{T}_2$ are quasi-uniform but need not align on the interface Γ.

Let $(\mathbf{V}_h, Q_{1,h})$ be a stable pair of mixed finite element spaces associated with the triangulation $\mathscr{T}_1$, for example, BDM [25], BDFM [24], or RT [26]. Also, let $Q_{2,h}$ be a conforming finite element space associated with $\mathcal{T}_2$. The functions in $Q_{2,h}$ vanish on $\partial\Omega_2\backslash\Gamma$. Then the discrete problem is as follows:

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_{1,h} \in Q_{1,h}$ and $p_{2,h} \in Q_{2,h}$ such that

$$
\begin{aligned}
(a^{-1}\mathbf{u}_h, \underline{\chi}) - (p_{1,h}, \nabla\cdot\underline{\chi}) + \langle p_{2,h}, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma &= \langle \mathbf{F}_0, \underline{\chi}\rangle \quad \text{for } \underline{\chi} \in \mathbf{V}_h \\
-(\nabla\cdot\mathbf{u}_h, q_1) &= \langle F_1, q_1\rangle \quad \text{for } q_1 \in Q_{1,h} \\
\langle \mathbf{u}_h\cdot\mathbf{n}_1, q_2\rangle_\Gamma - a(p_{2,h}, q_2) &= \langle F_2, q_2\rangle \quad \text{for } q_2 \in Q_{2,h}
\end{aligned}
\tag{10}
$$

As in the continuous case discussed above (see also Reference [9]), one groups together the spaces $\mathbf{V}_h$ and $Q_{2,h}$ and rewrites (10) in a matrix form similar to (8) in which the corresponding block operators are denoted $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{B}_h^{\mathrm{T}}$. It is immediate that $\mathbf{A}_h$ is coercive on $\operatorname{Ker}\mathbf{B}_h$ and the corresponding 'inf-sup' condition is exactly that required for the mixed approximation pair $(\mathbf{V}_h, Q_{1,h})$, i.e. there is a constant $C>0$, independent of h, so that

$$
C\|q_1\|_{0,\Omega_1} \leqslant \sup_{\underline{\chi}\in\mathbf{V}_h} \frac{(q_1, \nabla\cdot\underline{\chi})}{\|\underline{\chi}\|_{\mathbf{V}}} \quad \text{for all } q_1 \in Q_{1,h} \tag{11}
$$

The following result is an immediate consequence [24]:

Theorem 3.1. *The discrete problem* (10) *is uniquely solvable and if the finite element spaces* $(\mathbf{V}_h, Q_{1,h})$ *satisfy the inf–sup condition* (11) *then the following a priori estimate holds for its solution*:

$$
\|\mathbf{u}_h\|_{\mathbf{V}} + \|p_{1,h}\|_{0,\Omega_1} + \|p_{2,h}\|_{1,\Omega_2} \leqslant C(\|\mathbf{F}_0\|_{\mathbf{V}'} + \|F_1\|_{Q_1'} + \|F_2\|_{Q_2'}) \tag{12}
$$

The constant C is independent of the mesh sizes h_1 of $\mathcal{T}_1$ and h_2 of $\mathcal{T}_2$.

For the analysis, we shall need the L^2-projection operators $P_{i,h}: Q_i \mapsto Q_{i,h}$, $i=1,2$ and the approximation operator $\Pi_h : \mathbf{V}\cap\mathbf{H}^\gamma(\Omega_1) \mapsto \mathbf{V}_h$, $\gamma > 0$ associated with the mixed pair of subspaces. We assume that the operators $P_{1,h}$, $P_{2,h}$ and Π_h satisfy the following properties:

- (A.1) Π_h is a linear operator in $\mathbf{V}\cap\mathbf{H}^\gamma(\Omega_1)$, uniformly bounded with respect to h_1, and satisfies
  - (a) $\nabla\cdot\Pi_h\chi = P_{1,h}\nabla\cdot\underline{\chi}$ for all $\underline{\chi}\in\mathbf{V}$;
  - (b) if $\mathbf{v}\in\mathbf{V}$ satisfies $\mathbf{v}\cdot\mathbf{n}_1 = \mathbf{v}_h\cdot\mathbf{n}_1$ on Γ for some $\mathbf{v}_h\in\mathbf{V}_h$, then $(\Pi_h\mathbf{v})\cdot\mathbf{n}_1 = \mathbf{v}_h\cdot\mathbf{n}_1$ on Γ;
- (A.2) There is an integer $k > 0$ such that for all $\gamma\in(0,k]$:
  - (a) for all $\mathbf{u}\in\mathbf{V}\cap\mathbf{H}^\gamma(\Omega_1)$

$$
\|(I-\Pi_h)\mathbf{u}\|_{0,\Omega_1} \leqslant C \begin{cases} h_1^\gamma\|\mathbf{u}\|_{\gamma,\Omega_1} + h_1\|\operatorname{div}\mathbf{u}\| & \text{if } \gamma < 1, \\ h_1^\gamma\|\mathbf{u}\|_{\gamma,\Omega_1} & \text{if } \gamma \geqslant 1; \end{cases}
$$

  - (b) $\|(I-P_{1,h})p_1\|_{0,\Omega_1} \leqslant Ch_1^\gamma\|p_1\|_{\gamma,\Omega_1}$ for all $p_1\in H^\gamma(\Omega_1)$;
- (A.3) There is an integer $k > 0$ such that for all $\gamma\in(0,k]$:
  - (a) $\|(I-P_{2,h})p_2\|_{0,\Omega_2} \leqslant Ch_2^{1+\gamma}\|p_2\|_{1+\gamma,\Omega_2}$ for all $p_2\in H^{1+\gamma}(\Omega_2)\cap Q_2$;
  - (b) $\|(I-P_{2,h})p_2\|_{1,\Omega_2} \leqslant Ch_2^\gamma\|p_2\|_{1+\gamma,\Omega_2}$, for all $p_2\in H^{1+\gamma}(\Omega_2)\cap Q_2$.

Properties (A.1) and (A.2) are standard for the well-known mixed finite element spaces (BDM [25], BDFM [24], and RT [26]) and the associated operators Π_h and $P_{1,h}$. Identity (A.1) (a) ensures that the finite element pair $(\mathbf{V}_h, Q_{1,h})$ satisfies the 'inf–sup' condition (11), which in turn guarantees the stability of the mixed finite element method. Less known is the estimate (A.2)(a) for $\gamma < 1$. It can be proved by using the fact that the operator Π_h is bounded on the space $\mathbf{V} \cap \mathbf{H}^\gamma(\Omega_1)$ (see, e.g. [24, pp. 124-125]) and recovers element-wise constant vector fields. One can use these two properties of Π_h on a reference element $\hat{T}$, which leads to

$$\begin{aligned} \|(I-\Pi_h)\,\hat{\mathbf{u}}\|_{\hat{T}}^2 &= \|(I-\Pi_h)(\hat{\mathbf{u}}-\underline{c})\|_{\hat{T}}^2 \\ &\leqslant C\left[\inf_{\underline{c}} \|\,\hat{\mathbf{u}}-\underline{c}\|_{\gamma,\hat{T}}^2 + \|\mathrm{div}\ \hat{\mathbf{u}}\|_{\hat{T}}^2\right] \\ &\leqslant C(|\,\hat{\mathbf{u}}|_{\gamma,\hat{T}}^2 + \|\mathrm{div}\ \hat{\mathbf{u}}\|_{\hat{T}}^2). \end{aligned}$$

Here, $|.|_{\gamma,\hat{T}}$ stands for the $\mathbf{H}^\gamma(\hat{T})$ semi-norm. Now, changing variables back from the reference element to an element of diameter $\mathcal{O}(h_1)$, one arrives at the desired estimate (A.2)(a),

$$h_1^{-2}\|(I-\Pi_h)\mathbf{u}\|^2 \leqslant C(h_1^{-2+2\gamma}|\mathbf{u}|_{\gamma,\Omega_1}^2 + \|\mathrm{div}\,\mathbf{u}\|^2)$$

The approximation properties for the standard conforming Lagrangian finite element spaces are well known (cf. Reference [27]). The case of $\gamma = 0$ and (b) follows from [28].

The error analysis of the coupled method (10) is quite straightforward. We prove the following error estimate (stated in Reference [9]):

Theorem 3.2. *Let* $(\mathbf{u}, p_1, p_2)$ *and* $(\mathbf{u}_h, p_{1,h}, p_{2,h})$ *denote the solutions of* (7) *and* (10), *respectively. Let* $0 < \gamma \leqslant k$ *and assume that* $\mathbf{u} \in \mathbf{H}^\gamma(\Omega_1)$, $\nabla \cdot \mathbf{u}, p_1 \in H^\gamma(\Omega_1)$, *and* $p_2 \in H^{1+\gamma}(\Omega_2)$. *Then*

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}} + \|p_1 - p_{1,h}\|_{0,\Omega_1} + \|p_2 - p_{2,h}\|_{1,\Omega_2} \\ &\quad \leqslant C(h_1^\gamma\|\mathbf{u}\|_{\gamma,\Omega_1} + h_1^\gamma\|\nabla\cdot\mathbf{u}\|_{\gamma,\Omega_1} + h_1^\gamma\|p_1\|_{\gamma,\Omega_1} + h_2^\gamma\|p_2\|_{1+\gamma,\Omega_2}) \end{aligned} \tag{13}$$

with constant C independent of h_1 *and* h_2.

Proof
The approximation errors $\mathbf{e}_h = \Pi_h\mathbf{u} - \mathbf{u}_h$, $e_{1,h} = P_{1,h}p_1 - p_{1,h}$, and $e_{2,h} = P_{2,h}p_2 - p_{2,h}$ satisfy the discrete problem

$$\begin{aligned} (a^{-1}\mathbf{e}_h,\ \underline{\chi}) - (e_{1,h}, \nabla\cdot\underline{\chi}) + \langle e_{2,h}, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma &= \langle\mathbf{\Phi}_0, \underline{\chi}\rangle \quad \text{for } \underline{\chi}\in\mathbf{V}_h, \\ -(\nabla\cdot\mathbf{e}_h, q_1) &= 0 \quad \text{for } q_1 \in Q_{1,h} \\ \langle\mathbf{e}_h\cdot\mathbf{n}_1, q_2\rangle_\Gamma - a(e_{2,h}, q_2) &= \langle\Phi_2, q_2\rangle \quad \text{for } q_2 \in Q_{2,h} \end{aligned}$$

where

$$\langle\mathbf{\Phi}_0, \underline{\chi}\rangle = (a^{-1}(\Pi_h\mathbf{u}-\mathbf{u}), \underline{\chi}) + (P_{1,h}p_1 - p_1, \nabla\cdot\underline{\chi}) + \langle P_{2,h}p_2 - p_2, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma$$

and

$$\langle \Phi_2, q_2 \rangle = -a(P_{2,h} p_2 - p_2, q_2) + \langle (\Pi_h \mathbf{u} - \mathbf{u}) \cdot \mathbf{n}_1, q_2 \rangle_\Gamma.$$

We first introduce the space $H^{1/2}_{0,0}(\Gamma)$ as the interpolation space half-way between $H^1_0(\Gamma)$ and $L^2(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$. Then the cross terms (on Γ) are estimated based on standard trace inequalities valid for the spaces $H^{1/2}_{0,0}(\Gamma)$, the trace space of $H^1_0(\Omega_2, \partial\Omega_2 \backslash \Gamma)$ and for $H^{-1/2}(\Gamma)$ the normal trace of $\mathbf{H}(\text{div}, \Omega_1)$ (see, e.g., Reference [29]) as follows:

$$\begin{aligned} \langle P_{2,h} p_2 - p_2, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma &\leqslant \| P_{2,h} p_2 - p_2 \|_{H^{1/2}_{0,0}(\Gamma)} \| \underline{\chi} \cdot \mathbf{n}_1 \|_{-1/2,\,\Gamma} \\ &\leqslant C \| P_{2,h} p_2 - p_2 \|_{1,\ \Omega_2} \| \underline{\chi} \|_{\mathbf{V}} \end{aligned}$$

Similarly,

$$\begin{aligned} \langle (\Pi_h \mathbf{u} - \mathbf{u}) \cdot \mathbf{n}_1, q_2 \rangle_\Gamma &\leqslant \| q_2 \|_{H^{1/2}_{0,0}(\Gamma)} \| (\Pi_h \mathbf{u} - \mathbf{u}) \cdot \mathbf{n}_1 \|_{-1/2,\Gamma} \\ &\leqslant C \| \Pi_h \mathbf{u} - \mathbf{u} \|_{\mathbf{V}} \ \| q_2 \|_{1,\Omega_2} \end{aligned}$$

By the approximation properties (A.2) and (A.3) we have

$$|\langle \boldsymbol{\Phi}_0, \underline{\chi} \rangle| \leqslant C(h_1^\gamma \| \mathbf{u} \|_{\gamma,\Omega_1} + h_2^\gamma \| p_2 \|_{1+\gamma,\Omega_2}) \| \underline{\chi} \|_{\mathbf{V}}$$

and

$$\begin{aligned} |\langle \Phi_2, q_2 \rangle| &\leqslant C(h_2^\gamma \| p_2 \|_{1+\gamma,\Omega_2} + \| \Pi_h \mathbf{u} - \mathbf{u} \|_{\mathbf{V}}) \| q_2 \|_{1,\Omega_2} \\ &\leqslant C(h_2^\gamma \| p_2 \|_{1+\gamma,\Omega_2} + h_1^\gamma \| \mathbf{u} \|_{\gamma,\Omega_1} + h_1^\gamma \| \nabla \cdot \mathbf{u} \|_{\gamma,\Omega_1}) \| q_2 \|_{1,\Omega_2} \end{aligned}$$

The above two estimates and Theorem 3.1 show that $\| \mathbf{e}_h \|_{\mathbf{V}} + \| e_{1,h} \|_{0,\Omega_1} + \| e_{2,h} \|_{1,\Omega_2}$ is bounded by the right-hand side of (13). The theorem immediately follows from this estimate, the approximation properties of Π_h, $P_{1,h}$ and $P_{2,h}$, and the triangle inequality. ■

4. ITERATIVE SOLUTION

We now consider the problem of computing the solution of the coupled symmetric indefinite system (10) by preconditioning. The first preconditioner we shall discuss results from the *a priori* estimates for the discrete solution established in Theorem 3.1. The second preconditioner is derived by domain decomposition and is based on Poincaré–Steklov operators. This means that the solution of subdomain problems reduces to an iteration involving only unknowns on Γ.

4.1. Preconditioning the composite saddle-point problem

We first consider preconditioning the discrete algebraic system resulting from the composite problem. Let $\mathscr{X}$ denote the product space $\mathbf{V}_h \times Q_{1,h} \times Q_{2,h}$ and consider the operator $\mathscr{A} : \mathscr{X} \to \mathscr{X}$ given by

$$\mathscr{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{N}^{\mathrm{T}} & \mathbf{T}^{\mathrm{T}} \\ \mathbf{N} & 0 & 0 \\ \mathbf{T} & 0 & -A_2 \end{bmatrix} \tag{14}$$

Here

$$(\mathbf{A}_1 \underline{\chi}, \underline{\theta}) = (a^{-1} \underline{\chi}, \underline{\theta}) \quad \text{for all } \underline{\chi}, \underline{\theta} \in \mathbf{V}_h$$

$$(\mathbf{N} \underline{\chi}, q_1) = (\mathbf{N}^{\mathrm{T}} q_1, \underline{\chi}) = -(\nabla \cdot \underline{\chi}, q_1) \quad \text{for all } \underline{\chi} \in \mathbf{V}_h,\ q_1 \in Q_{1,h}$$

$$(\mathbf{T} \underline{\chi}, q_2) = (\mathbf{T}^{\mathrm{T}} q_2, \underline{\chi}) = \langle q_2, \underline{\chi} \cdot \mathbf{n}_1 \rangle_{\Gamma} \quad \text{for all } \underline{\chi} \in \mathbf{V}_h,\ q_2 \in Q_{2,h}$$

$$(A_2 q_2, r_2) = a(q_2, r_2) \quad \text{for all } q_2, r_2 \in Q_{2,h}.$$

We also consider the block diagonal operator

$$\mathscr{D} = \begin{bmatrix} \mathbf{\Lambda} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A_2 \end{bmatrix}$$

where $(\mathbf{\Lambda} \underline{\chi},\ \underline{\theta}) = (a^{-1} \underline{\chi},\ \underline{\theta}) + (\nabla \cdot \underline{\chi},\ \nabla \cdot \underline{\theta})$ for all $\underline{\chi},\ \underline{\theta} \in \mathbf{V}_h$. By Theorem 3.1 and the boundedness of $\mathscr{A}$, for any $\mathbf{U} \in \mathscr{X}$,

$$|||\mathbf{U}|||_{\mathscr{D}}^2 \leqslant C |||\mathscr{A}\mathbf{U}|||_{\mathscr{D}^{-1}}^2 = C \sup_{\mathbf{W} \in \mathscr{X}} \frac{(\mathscr{A}\mathbf{U}, \mathbf{W})^2}{(\mathscr{D}\mathbf{W}, \mathbf{W})} \leqslant C |||\mathbf{U}|||_{\mathscr{D}}^2 \tag{15}$$

Here $|||\cdot|||_{\mathscr{D}}$ denotes the operator norm given by $|||\cdot|||_{\mathscr{D}} = (\mathscr{D}\cdot, \cdot)^{1/2}$ and the pairing $(\cdot, \cdot)$ denotes the inner-product in the product space $\mathscr{X}$. Note that inverting $\mathscr{D}$ is equivalent to solving a mixed finite element problem in Ω_1 with homogeneous Dirichlet data on Γ and a Galerkin finite element problem in Ω_2 with homogeneous Neumann data on Γ.

In practice, one represents the above operators in terms of bases. Combining the bases for the three spaces which define $\mathscr{X}$ gives rise to a basis $\{\Psi_i\}_{i=1}^n$. Let $\bar{\mathscr{A}}$ be the matrix corresponding to the operator $\mathscr{A}$, i.e. $\bar{\mathscr{A}}_{ij} = (\mathscr{A}\Psi_j, \Psi_i)$. The matrix $\bar{\mathscr{D}}$ corresponding to $\mathscr{D}$ is defined analogously. The above inequality (15) can be rewritten in terms of matrices as

$$c_0 x^{\mathrm{T}} \bar{\mathscr{D}} x \leqslant x^{\mathrm{T}} (\bar{\mathscr{A}}^{\mathrm{T}} \bar{\mathscr{D}}^{-1} \bar{\mathscr{A}}) x \leqslant c_1 x^{\mathrm{T}} \bar{\mathscr{D}} x \tag{16}$$

for all $x \in \mathbf{R}^n$. Here $\bar{\mathscr{A}}^{\mathrm{T}}$ denotes the transpose of the matrix $\bar{\mathscr{A}}$. The algebraic problem corresponding to (10) is to find the vector $x \in \mathbf{R}^n$ satisfying $\bar{\mathscr{A}} x = b$ for an appropriately defined b.

Inequality (16) implies that reformulated system

$$\bar{\mathscr{A}}^{\mathrm{T}} \bar{\mathscr{D}}^{-1} \bar{\mathscr{A}} x = \bar{\mathscr{A}}^{\mathrm{T}} \bar{\mathscr{D}}^{-1} b$$

can be preconditioned by $\bar{\mathscr{D}}$. This can be solved by a rapidly convergent preconditioned iteration. In addition, the operators $\boldsymbol{\Lambda}$ and A_2 can be replaced by preconditioners. Instead of preconditioning the normal system one can alternatively precondition the original saddle-point system using the same block-diagonal preconditioner $\bar{\mathscr{D}}$ and apply the minimum residual method.

4.2. *Preconditioning the reduced problem by interface domain decomposition*

We next consider two strategies based on domain decomposition. Specifically, we consider the case when software is available for solving the mixed and finite element problems independently. The idea is to reduce the original problem (10) to a problem on Γ. We give two examples of such reductions. The reduced problems are symmetric, positive definite and well conditioned with respect to appropriate inner-products.

To derive the reduced system, we first introduce $(\tilde{\mathbf{u}}_h, \tilde{p}_{1,h}, \tilde{p}_{2,h})$ in $\mathbf{V}_h \times Q_{1,h} \times Q_{2,h}$ satisfying

$$\begin{aligned}(a^{-1}\tilde{\mathbf{u}}_h, \underline{\chi}) - (\tilde{p}_{1,h}, \nabla\cdot\underline{\chi}) &= \langle \mathbf{F}_0, \underline{\chi}\rangle \quad \text{for } \underline{\chi}\in\mathbf{V}_h\\ -(\nabla\cdot\tilde{\mathbf{u}}_h, q_1) &= \langle F_1, q_1\rangle \quad \text{for } q_1\in Q_{1,h}\\ -a(\tilde{p}_{2,h}, q_2) &= \langle F_2, q_2\rangle \quad \text{for } q_2\in Q_{2,h}\end{aligned} \tag{17}$$

These three equations represent two independent problems. The first two equations correspond to a mixed finite element problem on Ω_1, while the third is a Galerkin finite element problem on Ω_2. Once $\tilde{\mathbf{u}}_h$ and $\tilde{p}_{1,h}$ are computed, the solution $(\mathbf{u}, p_{1,h}, p_{2,h})$ of problem (10) can be represented in the form $(\mathbf{u}_h, p_{1,h}, p_{2,h}) = (\mathbf{v}_h + \tilde{\mathbf{u}}_h, r_{1,h} + \tilde{p}_{1,h}, r_{2,h} + \tilde{p}_{2,h})$ where the remainder $(\mathbf{v}_h, r_{1,h}, r_{2,h})$ satisfies the system

$$\begin{aligned}(a^{-1}\mathbf{v}_h, \underline{\chi}) - (r_{1,h}, \nabla\cdot\underline{\chi}) &= -\langle \tilde{p}_{2,h} + r_{2,h}, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma \quad \text{for } \underline{\chi}\in\mathbf{V}_h\\ -(\nabla\cdot\mathbf{v}_h, q_1) &= 0 \quad \text{for } q_1\in Q_{1,h}\\ -a(r_{2,h}, q_2) &= -\langle(\tilde{\mathbf{u}}_h + \mathbf{v}_h)\cdot\mathbf{n}_1, q_2\rangle_\Gamma \quad \text{for } q_2\in Q_{2,h}\end{aligned} \tag{18}$$

which is coupled only through the boundary values of $r_{2,h}$ and $\mathbf{v}_h\cdot\mathbf{n}_1$ on Γ.

We will reformulate (18) in terms of operators on the trace spaces on Γ. To this end, we introduce the discrete trace spaces

$$Q_h^{-1/2}(\Gamma) = \{\underline{\chi}\cdot\mathbf{n}_1 \text{ on } \Gamma : \underline{\chi}\in\mathbf{V}_h\}$$

and

$$Q_h^{1/2}(\Gamma) = \{q|_\Gamma : q\in Q_{2,h}\}$$

which are obviously subspaces of $H^{-1/2}(\Gamma)$ and $H^{1/2}_{0,0}(\Gamma)$, respectively, and inherit their norms.

We now define the operator

$$E : Q_h^{1/2}(\Gamma) \mapsto Q_h^{-1/2}(\Gamma) \quad \text{by } E\lambda = \mathbf{v}_h(\lambda)\cdot\mathbf{n}_1 \quad \text{on } \Gamma \tag{19}$$

where for a given $\lambda \in Q_h^{1/2}(\Gamma)$ the pair $(\mathbf{v}_h(\lambda), s_{1,h}(\lambda)) \in \mathbf{V}_h \times Q_{1,h}$ is the solution of the problem

$$\begin{aligned} (a^{-1}\mathbf{v}_h(\lambda), \underline{\chi}) - (s_{1,h}(\lambda), \nabla \cdot \underline{\chi}) &= \langle \lambda, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma \quad \text{for } \underline{\chi} \in \mathbf{V}_h \\ -(\nabla \cdot \mathbf{v}_h(\lambda), q_1) &= 0 \quad \text{for } q_1 \in Q_{1,h} \end{aligned} \tag{20}$$

The operator E has a meaning of a discrete Dirichlet–Neumann mapping on Γ.

Further, we define the second operator

$$S : Q_h^{-1/2}(\Gamma) \mapsto Q_h^{1/2}(\Gamma) \quad \text{by } S\lambda^* = s_{2,h}(\lambda^*) \quad \text{on } \Gamma \tag{21}$$

where for a given $\lambda^* \in Q_h^{-1/2}(\Gamma)$ the function $s_{2,h}(\lambda^*) \in Q_{2,h}$ is the solution of

$$a(s_{2,h}(\lambda^*), q_2) = \langle \lambda^*, q_2 \rangle_\Gamma \quad \text{for all } q_2 \in Q_{2,h} \tag{22}$$

Clearly, $s_{2,h}(\lambda^*)$ is discrete harmonic. This is the Neumann–Dirichlet mapping on Γ.

The boundness of both operators S and E follows from the stability of the corresponding boundary-value problems with respect to the boundary data and the following trace inequalities:

$$\|q_2\|_{H_{0,0}^{1/2}(\Gamma)} \leqslant C\|q_2\|_{1,\Omega_2} \quad \text{for all } q_2 \in Q_2 \tag{23}$$

and

$$\|\underline{\eta} \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} \leqslant \|\underline{\eta} \cdot \mathbf{n}\|_{H^{-1/2}(\partial\Omega_1)} \leqslant C\|\underline{\eta}\|_{\mathbf{V}} \quad \text{for all } \underline{\eta} \in \mathbf{V} \tag{24}$$

Note, that the bounds do not depend on h_1 or h_2.

In terms of the operators S and E, (18) becomes

$$\begin{aligned} \mathbf{v}_h \cdot \mathbf{n}_1 &= -E(\tilde{p}_{2,h}^\Gamma + r_{2,h}^\Gamma) \\ r_{2,h}^\Gamma &= S((\tilde{\mathbf{u}}_h + \mathbf{v}_h) \cdot \mathbf{n}_1) \end{aligned} \tag{25}$$

Here $\tilde{p}_{2,h}^\Gamma$ and $r_{2,h}^\Gamma$ are the traces of $\tilde{p}_{2,h}$ and $r_{2,h}$ on Γ, respectively.

We can now further reduce the problem to a problem for either the trace $r_{2,h}^\Gamma$ or the trace $\mathbf{v}_h \cdot \mathbf{n}_1$. Namely, elimination of $\mathbf{v}_h \cdot \mathbf{n}_1$ gives

$$(I + SE)r_{2,h}^\Gamma = S(\tilde{\mathbf{u}}_h \cdot \mathbf{n}_1 - E\tilde{p}_{2,h}^\Gamma) \tag{26}$$

while elimination of $r_{2,h}^\Gamma$ leads to the problem

$$(I + ES)(\mathbf{v}_h \cdot \mathbf{n}_1) = -E(\tilde{p}_{2,h}^\Gamma + S(\tilde{\mathbf{u}}_h \cdot \mathbf{n}_1)) \tag{27}$$

Note that one can immediately recover the remainder $(\mathbf{v}_h, r_{1,h}, r_{2,h})$ from $r_{2,h}^\Gamma$ or from $\mathbf{v}_h \cdot \mathbf{n}_1$ by one additional solve on each subdomain. Thus, (26) reduces problem (25) of finding the trace $r_{2,h}^\Gamma$ on the boundary Γ of the remainder $r_{2,h}$, while problem (27) reduces to finding the trace $\mathbf{v}_h \cdot \mathbf{n}_1$ on the boundary Γ.

First, we shall study the properties of problem (26). We consider the inner product on $Q_h^{1/2}(\Gamma) \times Q_h^{1/2}(\Gamma)$ defined by

$$\langle\langle \lambda, \mu \rangle\rangle = a(\bar{s}_{2,h}(\lambda), \bar{s}_{2,h}(\mu)) \quad \text{for } \lambda, \mu \in Q_h^{1/2}(\Gamma) \tag{28}$$

where $\bar{s}_{2,h}(\lambda)$ and $\bar{s}_{2,h}(\mu)$ denote the discrete harmonic extensions of the Dirichlet data λ and μ, respectively, in the space $Q_{2,h}$, i.e. $\bar{s}_{2,h}(\lambda)=\lambda$ on Γ and

$$a(\bar{s}_{2,h}(\lambda), q_2)=0 \quad \text{for all } q_2 \in Q_{2,h},\ q_2=0 \text{ on } \Gamma \tag{29}$$

Note that the extension defined by (22) takes Neumann data $\lambda^* \in Q_h^{-1/2}(\Gamma)$ and produces discrete harmonic function $s_{2,h}(\lambda^*)$, while the extension defined by (29) takes Dirichlet data $\lambda \in Q_h^{1/2}(\Gamma)$ and produces a discrete harmonic function $\bar{s}_{2,h}(\lambda)$. It is well known that the above inner product introduces a norm on $Q_h^{1/2}(\Gamma)$ which is equivalent to the $H_{0,0}^{1/2}(\Gamma)$ norm. This equivalence holds uniformly in h_2. The following theorem shows that (26) can be effectively solved by the conjugate gradient method.

Theorem 4.1. *The operator SE is symmetric and positive semi-definite with respect to the inner product $\langle\langle\cdot,\cdot\rangle\rangle$. Moreover, SE is bounded in the corresponding norm with bound K independent of h_1 and h_2. Thus, $(I + SE)$ is symmetric and positive definite on $Q_h^{1/2}(\Gamma)$ and has a condition number bounded by $K + 1$. The resulting conjugate gradient method converges with a rate bounded independent of h_1 and h_2.*

Proof
Let λ be in $Q_h^{1/2}(\Gamma)$. Then $S\lambda$ is in $Q_h^{-1/2}(\Gamma)$ and by definition (21) $SE\lambda=s_{2,h}(E\lambda)$ and is discrete harmonic. Thus, for any $\mu \in Q_h^{1/2}(\Gamma)$,

$$\langle\langle SE\lambda, \mu\rangle\rangle = a(s_{2,h}(E\lambda), \bar{s}_{2,h}(\mu)) = \langle E\lambda, \mu\rangle_\Gamma$$

Here $\bar{s}_{2,h}(\mu)$ is the discrete harmonic extension of the Dirichlet data μ and the last equality follows from definition (22) of $s_{2,h}$. Since $E\lambda=\mathbf{v}_h(\lambda)\cdot\mathbf{n}_1$, then using the fact that $(\nabla\cdot\mathbf{v}_h(\lambda), q_1)=0$ for all $q_1 \in Q_{1,h}$ and taking $\underline{\chi}=\mathbf{v}_h(\mu)$ to be the solution of (20) for the data $\lambda=\mu$ we get

$$\langle\langle SE\lambda, \mu\rangle\rangle = (a^{-1}\mathbf{v}_h(\lambda), \mathbf{v}_h(\mu))$$

This shows that SE is symmetric and positive semi-definite.

Finally, it easily follows from (24) and the stability properties of the mixed finite element problem on Ω_1 that

$$\langle\langle SE\lambda, \lambda\rangle\rangle = (a^{-1}\mathbf{v}_h(\lambda), \mathbf{v}_h(\lambda)) \leqslant C\|\lambda\|^2_{H_{0,0}^{1/2}(\Gamma)}.$$

The theorem follows from the equivalence of the norm $\langle\langle\cdot,\cdot\rangle\rangle^{1/2}$ with the $H_{0,0}^{1/2}(\Gamma)$ norm on $Q_h^{1/2}(\Gamma)$. ∎

Remark 4.1
The actions of the operator E are computed by solving the corresponding mixed finite element problem posed on Ω_1 with specified Dirichlet boundary conditions on Γ. The operator S corresponds to solving a (standard Galerkin finite element) problem posed on Ω_2 with specified Neumann boundary conditions on Γ.

Remark 4.2
The inner product which makes $I+SE$ into a symmetric and positive-definite operator involves discrete harmonic extension with respect to the subspace $Q_{2,h}$. This poses no additional computational problems. In fact, a carefully implemented conjugate gradient algorithm for (26) need only have one mixed solve on Ω_1 and one finite element solve on Ω_2 per iterative step (after startup).

We now turn our attention to problem (27). Given $\lambda^* \in Q_h^{-1/2}(\Gamma)$, introduce the 'discrete mixed harmonic' extension $(\bar{\mathbf{u}}_h(\lambda^*), \bar{p}_{1,h}(\lambda^*)) \equiv (\bar{\mathbf{u}}_h, \bar{p}_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$ as a solution to the problem

$$
\begin{gathered}
\bar{\mathbf{u}}_h \cdot \mathbf{n}_1 = \lambda^* \text{ on } \Gamma \\
(a^{-1}\bar{\mathbf{u}}_h, \underline{\chi}) - (\bar{p}_{1,h}, \nabla \cdot \underline{\chi}) = 0 \quad \text{for all } \underline{\chi} \text{ in } \mathbf{V}_h \text{ with } \underline{\chi} \cdot \mathbf{n}_1 = 0 \text{ on } \Gamma \\
(\nabla \cdot \bar{\mathbf{u}}_h, q_1) = 0 \quad \text{for all } q_1 \text{ in } Q_{1,h}
\end{gathered} \tag{30}
$$

Let $\langle\langle \cdot,\cdot \rangle\rangle_*$ denote the following inner product in $Q_h^{-1/2}(\Gamma)$:

$$
\langle\langle \lambda^*, \mu^* \rangle\rangle_* = (a^{-1}\bar{\mathbf{u}}_h(\lambda^*), \bar{\mathbf{u}}_h(\mu^*)) \tag{31}
$$

where $\bar{\mathbf{u}}_h(\lambda^*)$ and $\bar{\mathbf{u}}_h(\mu^*)$ are the 'discrete mixed harmonic' extensions of λ^* and μ^* defined by (30).

Lemma 4.1. *For a quasi-uniform mesh partitions $\mathscr{T}_1$ the inner product defined by* (31) *gives rise to a norm which is equivalent* (*independent of h_1*) *to $\|\cdot\|_{H^{-1/2}(\Gamma)}$ on $Q_h^{-1/2}(\Gamma)$. That is, there are constants c and C, independent of h, such that*

$$
c\|\lambda^*\|^2_{H^{-1/2}(\Gamma)} \leqslant \langle\langle \lambda^*, \lambda^* \rangle\rangle_* \leqslant C\|\lambda^*\|^2_{H^{-1/2}(\Gamma)} \quad \textit{for all } \lambda^* \in Q_h^{-1/2}(\Gamma)
$$

Proof
By definition, for $\lambda^* \in Q_h^{-1/2}(\Gamma)$ there is an $\mathbf{u}_h \in \mathbf{V}_h$ such that $\lambda^* = \mathbf{u}_h \cdot \mathbf{n}_1$.
Then the lower bound

$$
\|\lambda^*\|^2_{H^{-1/2}(\Gamma)} = \|\mathbf{u}_h \cdot \mathbf{n}_1\|^2_{H^{-1/2}(\Gamma)} \leqslant C(a^{-1}\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h)
$$

follows immediately from (24) and (30).

For the other bound, we reduce the problem to one of discrete (divergence-free) extension. Suppose that we have defined $\hat{\mathbf{u}}_h \in \mathbf{V}_h$ with $\hat{\mathbf{u}}_h \cdot \mathbf{n}_1 = \lambda^*$ on Γ, $\nabla \cdot \hat{\mathbf{u}}_h = 0$ in Ω_1 and

$$
\|\hat{\mathbf{u}}_h\|_{\mathbf{V}} \leqslant C\|\lambda^*\|_{H^{-1/2}(\Gamma)} \tag{32}
$$

Let $_h = \hat{\mathbf{u}}_h + \mathbf{z}_h$. Then by (30) $\mathbf{z}_h \cdot \mathbf{n}_1 = 0$ on Γ and

$$
\begin{gathered}
(a^{-1}\mathbf{z}_h, \underline{\chi}) - (\nabla \cdot \underline{\chi}, \bar{p}_{1,h}) = -(a^{-1}\hat{\mathbf{u}}_h, \underline{\chi}) \quad \text{for all } \underline{\chi} \text{ in } \mathbf{V}_h \quad \text{with } \underline{\chi} \cdot \mathbf{n}_1 = 0 \text{ on } \Gamma \\
(\nabla \cdot \mathbf{z}_h, q_1) = 0 \quad \text{for all } q_1 \text{ in } Q_{1,h}
\end{gathered}
$$

Since the mixed finite element pair is stable, therefore an *a priori* estimate holds and based on (32), we get

$$\|\mathbf{z}_h\|_{\mathbf{V}} \leqslant C\|\,\hat{\mathbf{u}}_h\|_{\mathbf{V}} \leqslant C\|\lambda^*\|_{H^{-1/2}(\Gamma)}$$

It is then immediate from the triangle inequality that

$$(a^{-1}\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h)^{1/2} \leqslant C\|\bar{\mathbf{u}}_h\|_{\mathbf{V}} \leqslant C(\|\,\hat{\mathbf{u}}_h\|_{\mathbf{V}} + \|\mathbf{z}_h\|_{\mathbf{V}}) \leqslant C\|\lambda^*\|_{H^{-1/2}(\Gamma)}$$

This verifies the second inequality of the lemma.

Thus, to complete the proof of the lemma, we need only to construct $\hat{\mathbf{u}}_h$ satisfying (32). A proof for two-dimensional domains was given in Reference [30] and with this the proof of the lemma is complete.

We provide an alternative proof which relies on elliptic regularity. This proof remains valid for higher-dimensional applications.

We consider the function ϕ satisfying

$$\Delta\phi = 0 \quad \text{in } \Omega_1, \quad \frac{\partial\phi}{\partial\mathbf{n}_1} = \mathbf{u}_h\cdot\mathbf{n}_1 \quad \text{on } \Gamma, \;\; \phi = 0 \quad \text{on } \partial\Omega_1\backslash\Gamma$$

Then

$$D(\phi,\theta) \equiv (\nabla\phi, \nabla\theta) = F(\theta) \tag{33}$$

for all $\theta \in H^1(\Omega_1)$ with $\theta = 0$ on $\partial\Omega_1\backslash\Gamma$. Here $D(\cdot,\cdot)$ is the Dirichlet inner product on Ω_1 and F denotes the functional $F(\theta) = \langle \mathbf{u}_h\cdot\mathbf{n}_1, \theta\rangle_\Gamma$. We define $\hat{\mathbf{u}}_h = \Pi_h(\nabla\phi)$.

Clearly $(\nabla\phi)\cdot\mathbf{n}_1 = \mathbf{u}_h\cdot\mathbf{n}_1$ on Γ and so, by (A.1) (b), we have $(\Pi_h\nabla\phi)\cdot\mathbf{n}_1 = \mathbf{u}_h\cdot\mathbf{n}_1$ on Γ.

Furthermore,

$$\|\nabla\phi\|_{\mathbf{V}} \leqslant C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}$$

In addition, for any γ in $(0, 1/2)$,

$$\|(I - \Pi_h)(\nabla\phi)\|_{\mathbf{V}} \leqslant Ch_1^\gamma\|\phi\|_{1+\gamma,\Omega_1}, \quad \phi\in H^{1+\gamma}(\Omega_1)$$

For some γ in $(0, 1/2)$, the following regularity estimate holds for the mixed boundary value problem: Solutions of (33) satisfy

$$\|\phi\|_{1+\gamma,\Omega_1} \leqslant C(\gamma)\|F\|_{-1+\gamma,\Omega_1}$$

Now

$$\|F\|_{-1+\gamma,\Omega_1} = \sup_\theta \frac{\langle\mathbf{u}_h\cdot\mathbf{n}_1,\theta\rangle_\Gamma}{\|\theta\|_{1-\gamma,\Omega_1}} \leqslant C\sup_\theta \frac{\langle\mathbf{u}_h\cdot\mathbf{n}_1,\theta\rangle_\Gamma}{\|\theta\|_{1/2-\gamma,\Gamma}} = C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2+\gamma}(\Gamma)}$$

Here the supremum is over θ in $H^1(\Omega_1)$ with $\theta = 0$ on $\partial\Omega_1\backslash\Gamma$. Since the mesh in Ω_1 is quasi-uniform we get

$$\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2+\gamma}(\Gamma)} \leqslant Ch_1^{-\gamma}\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

Combining the above inequalities gives

$$\|\,\hat{\mathbf{u}}_h\|_{\mathbf{V}} \leqslant \|\nabla\phi\|_{\mathbf{V}} + \|(I-\Pi_h)\nabla\phi\|_{\mathbf{V}} \leqslant C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

This completes the proof of the lemma. ■

Theorem 4.2. *The operator ES is symmetric and positive semi-definite with respect to the inner product* $\langle\langle\cdot,\cdot\rangle\rangle_*$ *defined by* (31). *Moreover, for quasi-uniform meshes ES is bounded in the corresponding norm with bound K independent of* h_1 *and* h_2. *Thus,* $(I+ES)$ *is symmetric and positive definite on* $Q_h^{-1/2}(\Gamma)$ *and has a condition number bounded by* $K+1$. *The resulting conjugate gradient iteration converges with a rate bounded independent of* h_1 *and* h_2.

Proof
By the definition of the inner product $\langle\langle\cdot,\cdot\rangle\rangle_*$

$$\langle\langle ES\lambda^*, \mu^*\rangle\rangle_* = (a^{-1}\bar{\mathbf{u}}_h(ES\lambda^*), \bar{\mathbf{u}}_h(\mu^*)) \quad \text{for } \lambda^*, \mu^* \in Q_h^{-1/2}(\Gamma)$$

From the definitions of the operator E and the 'discrete mixed harmonic' extension $\bar{\mathbf{u}}_h(ES\lambda^*)$ one has $\bar{\mathbf{u}}_h(ES\lambda^*)\cdot\mathbf{n}_1 = ES\lambda^* = \mathbf{v}_h(S\lambda^*)\cdot\mathbf{n}_1$ on Γ, where $\mathbf{v}_h(S\lambda^*)$ is defined by (20) and is 'discrete mixed harmonic' in Ω_1, that is, $\mathbf{v}_h(S\lambda^\star) = \bar{\mathbf{u}}_h(ES\lambda^\star)$. Therefore, from the definition of S and from (20) with $\underline{\chi} = \bar{\mathbf{u}}_h(\mu^*)$ we get

$$(a^{-1}\bar{\mathbf{u}}_h(ES\lambda^*), \bar{\mathbf{u}}_h(\mu^*)) = \langle S\lambda^*, \mu^*\rangle_\Gamma = \langle s_{2,h}(\lambda^*), \mu^*\rangle_\Gamma = a(s_{2,h}(\lambda^*), s_{2,h}(\mu^*))$$

where for the last equality we have used (22) with $q_2 = s_{2,h}(\lambda^*)$. This shows that ES is symmetric and positive semi-definite. Thus, the theorem is a consequence of the *a priori* estimate

$$\|s_{2,h}(\lambda^*)\|_{1,\Omega_2} \leqslant C\|\lambda^*\|_{H^{-1/2}(\Gamma)}$$

and Lemma 4.1. ■

5. NUMERICAL EXPERIMENTS

We present numerical experiments that illustrate the accuracy of the coupled approximation for smooth solutions and the convergence of the proposed iterative methods. An L-shaped domain is used only to generate easily genuinely non-matching grids along the subdomain interfaces. Namely, we consider the following two-dimensional test problem:

- the domain is $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Omega_1 = (0,1)\times(0,1)$, $\Gamma = \{(1,y),\ 0<y<b\}$, $b<1$ is a given parameter, and $\Omega_2 = (1, 1+b)\times(0,b)$;
- the elliptic problem in Ω_1 is $-\nabla\cdot a_1\nabla p_1 = f_1$, where the coefficient matrix

  $$a_1 = \begin{bmatrix} 1+10x^2+y^2 & \frac{1}{2}+x^2+y^2 \\ \frac{1}{2}+x^2+y^2 & 1+x^2+10y^2 \end{bmatrix}$$

  the exact solution is $p_1(x,y) = (1-x)^2x(1-y)y$, hence $\mathbf{u} = -a_1\nabla p_1$.
- the elliptic problem in Ω_2 is $-\nabla\cdot a_2\nabla p_2 = f_2$, where the coefficient matrix is just the identity, i.e. $a_2 = I$, and the exact solution is $p_2(x,y) = 10^5(1+b-x)(x-1)^2y(b-y)$.

This choice of the domain allowed us to easily generate truly non-matching hierarchical grids. The experiments we did for $b=1$ (i.e. on a rectangular domain) were slightly better.

Note that,

$$p(x,y) = \begin{cases} p_1(x,y) & \text{in } \Omega_1 \\ p_2(x,y) & \text{in } \Omega_2 \end{cases}$$

is an $H^1(\Omega)$–function since $[p]|_\Gamma = 0$ and $(a_1\nabla p_1)\cdot \mathbf{n}_1 = (a_2\nabla p_1)\cdot \mathbf{n}_1$ on Γ. Also, p vanishes on $\partial\Omega$.

To discretize the problem we used lowest order Raviart–Thomas spaces on uniform triangular mesh of size h_1 in Ω_1 and conforming piecewise linear functions over uniform triangles in Ω_2 with mesh-size h_2. We write the resulting linear system in the form

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{N}^{\mathrm{T}} & \mathbf{T}^{\mathrm{T}} \\ \mathbf{N} & 0 & 0 \\ \mathbf{T} & 0 & -A_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_{1,h} \\ p_{2,h} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{34}$$

We used the following solution methods:

(a) The minimum residual method (MINRES) for system (34) with the preconditioner

$$\begin{bmatrix} \mathbf{B}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A_2 \end{bmatrix}$$

Here, $\mathbf{B}_1$ stands for an algebraically stabilized version of the hierarchical basis method (HB) from Reference [31]. Details on the algebraic stabilization of the hierarchical base methods are found in Reference [32]. The I is the (diagonal) mass matrix and A_2 is the stiffness matrix for the finite element problem on Ω_2, which we invert exactly. This corresponds to solving a discrete Neumann problem in Ω_2. An alternative multigrid preconditioner for $\mathbf{A}_1$ can be found in Reference [33].

(b) The conjugate gradient method applied to the reduced problem (26). The stopping criterion here was until relative residual reduction of 10^{-6} was reached. The implementation of this method requires the action of the Schur complement $\langle \tilde{S}\tilde{w}_2, \tilde{v}_2\rangle_\Gamma = (a_2\nabla\tilde{w}_2,\ \nabla\tilde{v}_2)$. The matrix $\tilde{S}$ can be computed by LU factorization. The resulting matrix can also be used to compute the inner product $\langle\langle\lambda,\mu\rangle\rangle = \mu^{\mathrm{T}}\tilde{S}\lambda$. Alternatively, the computation of $\tilde{S}$ can be avoided with careful implementation (although one will have to solve one Neumann problem for each step in the iteration). The action of E is computed by inner iterations applied to solve the corresponding mixed problems (with a high accuracy) on Ω_1.

Let the grid in Ω_1 have mesh-nodes denoted by (x_i, y_j), $0 \leqslant i \leqslant n_x$, $0 \leqslant j \leqslant n_y$, $n_x = n_y = 1/h_1$, $h_x = h_y := h_1$. For computing the errors we have used shifted by half step-size points, namely we use the points $x_{i-1/2} = x_i - 0.5h_x$ and $y_{j-1/2} = y_j - 0.5h_y$. Finally, I_h stands for the finite element interpolation operator.

In each row of Table I we show:

(i) $$\delta_{p_1} \equiv \left\{ \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} h_x h_y [(p_1(x_{i-2/3}, y_{j-1/3}) - p_{1,h}(x_{i-2/3}, y_{j-1/3}))^2 + (p_1(x_{i-1/3}, y_{j-2/3}) - p_{1,h}(x_{i-1/3}, y_{j-2/3}))^2] \right\}^{1/2}$$

i.e. a discrete L^2-norm of the error $p_1 - p_{1,h}$;

(ii) $$\delta_{u_1} \equiv \left[\sum_{i=0}^{n_x}\sum_{j=1}^{n_y} h_x h_y (u_1(x_i, y_{j-1/2}) - u_{h,1}(x_i, y_{j-1/2}))^2 \right]^{1/2}$$

Table I. Error behaviour and iteration counts for the composite problem; $b = 0.55$ for iteration MINRES.

	$h_1 = 1/16$ $h_2 = b/16$	$h_1 = 1/32$ $h_2 = b/32$	$h_1 = 1/64$ $h_2 = b/64$	$h_1 = 1/128$ $h_2 = b/128$	$\approx$ order
δ_{p_1}	3.18e-2	7.57e-3	1.83e-3	4.57e-4	2
δ_{u_1}	0.5749	1.34e-1	3.27e-2	7.87e-3	2
δ_{u_2}	0.3617	8.87e-2	2.21e-2	5.51e-3	2
$\delta_{u_{\mathrm{int}}}$	0.3792	9.42e-2	2.37e-2	5.93e-3	2
δ_{p_2}	0.1519	3.44e-2	7.71e-3	1.91e-3	2
# iterations	57	71	86	92	
ϱ	0.69	0.74	0.78	0.79	

i.e. a discrete L^2-norm of the error $u_1 - u_{h,1}$. Here and in (ii) we used the respective components u_1 and u_2 of $\mathbf{u} = (u_1, u_2)$;

(iii) $$\delta_{u_2} = \|I_h u_2 - u_{h,2}\|_h \equiv \left[\sum_{i=1}^{n_x}\sum_{j=0}^{n_y} h_x h_y (u_2(x_{i-1/2}, y_j) - u_{h,2}(x_{i-1/2}, y_j))^2\right]^{1/2}$$

i.e. a discrete L^2-norm of the error $u_2 - u_{h,2}$;

(iv) $$\delta_{u_{\mathrm{int}}} \equiv \left[\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} h_x h_y ((\mathbf{u}\cdot\mathbf{n})(x_{i-1/2}, y_{j-1/2}) - (\mathbf{u}_h\cdot\mathbf{n})(x_{i-1/2}, y_{j-1/2}))^2\right]^{1/2}$$

i.e. a discrete L^2-norm of the error $\mathbf{u}\cdot\mathbf{n} - \mathbf{u}_h\cdot\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector to the edge with end-points (x_{i-1}, y_{j-1}) and (x_i, y_j); one should note that $\delta_{u_1}, \delta_{u_2}, \delta_{u_{\mathrm{int}}}$ are the summed flux error per triangle. Obviously, the sum of these squared errors will produce the squared discrete L^2-norm of the flux error.

(v) δ_{p_2}, a discrete $H^{1/2}_{0,0}(\Gamma)$-norm of the error $(I_h p_2 - p_{2,h})|_\Gamma$;

(vi) the number of iterations of the preconditioned MINRES method;

(vii) an average reduction factor ϱ.

Note the good accuracy of the composite discretization seen in the last column of Table I. The composite method preserves the error behaviour of the individual (subdomain) discretization methods which is in agreement with Theorem 3.2. In fact, the last column of Table I shows that both the pressure and the normal fluxes are superconvergent at the point they have been computed. This fact is well known for pressure (see, e.g. Reference [24, chapter V]). Since our solution is smooth and the mortar space ensures second order approximation it is not surprising that the computations show superconvergence in the pressure for the mortar method as well.

The seemingly unsatisfactory convergence of the preconditioned MINRES method is due to the quality of the $\mathbf{H}(\mathrm{div})$ preconditioner $\mathbf{B}_1$.

The second test demonstrates the convergence of the CG method applied to the matrix of the reduced problem (26). We have chosen a random $rhs_{2,h}$ and the iterations were stopped after reducing the norm of the residual by 10^{-6}. Here we varied the meshes h_1 and h_2 to see the sensitivity of the method with respect to the discrepancy of the grids (Table II). The convergence appears to be fairly insensitive to the mesh sizes, all in good agreement with the theory (see, e.g., Theorem 4.1).

Table II. Number of CG iterations and average reduction factors for solving the system $(I + SE)q_{2,h} = rhs_{2,h}$; $b = 0.55$.

	h_2			
h_1	$b/16$	$b/32$	$b/64$	$b/128$
1/16	11, 0.21	12, 0.26	13, 0.30	13, 0.30
1/32	12, 0.30	15, 0.39	15, 0.39	15, 0.39
1/64	10, 0.22	14, 0.36	16, 0.39	15, 0.39
1/128	9, 0.21	11, 0.27	15, 0.38	16, 0.40

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